SMILE WORKCARDS

Combined Transformations Pack Two

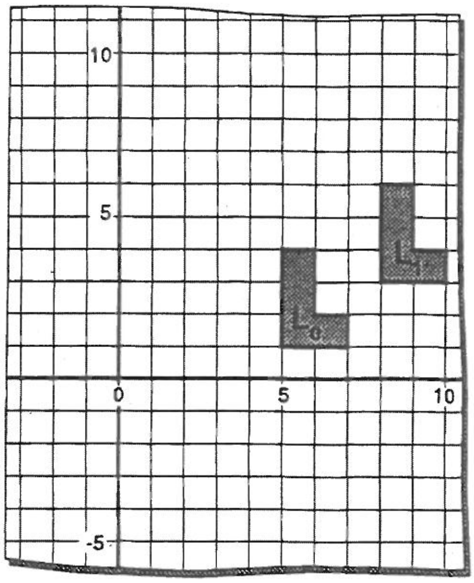
Contents

Combining Transformations

- Draw a grid with *x*-axis from -7 to 14 and *y*-axis from -7 to 10.
  Plot the points (5, 1), (7, 1), (7, 2), (6, 2), (6, 4), (5, 4), (5, 1) and join them in order.
  Shade the 'L' shape and label it L_0.

- Draw the resulting 'L' shapes for the transformations in the table below.
  The first transformed 'L' shape L_1 has been completed for you.

Starting Shape	Transformation	Label of new shape
L_0	translate $\begin{pmatrix} 3 \\ 2 \end{pmatrix}$	L_1
L_1	translate $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$	L_2
L_0	reflect in *x*-axis	L_3
L_3	reflect in *y*-axis	L_4
L_0	reflect in y = x	L_5
L_5	rotate 180° about (0,0) anticlockwise	L_6
L_0	rotate 90° about (0, 0) anticlockwise	L_7
L_7	rotate 180° about (0, 0)	L_8
L_0	reflect in $x = 3$	L_9
L_9	reflect in $x = -2$	L_{10}



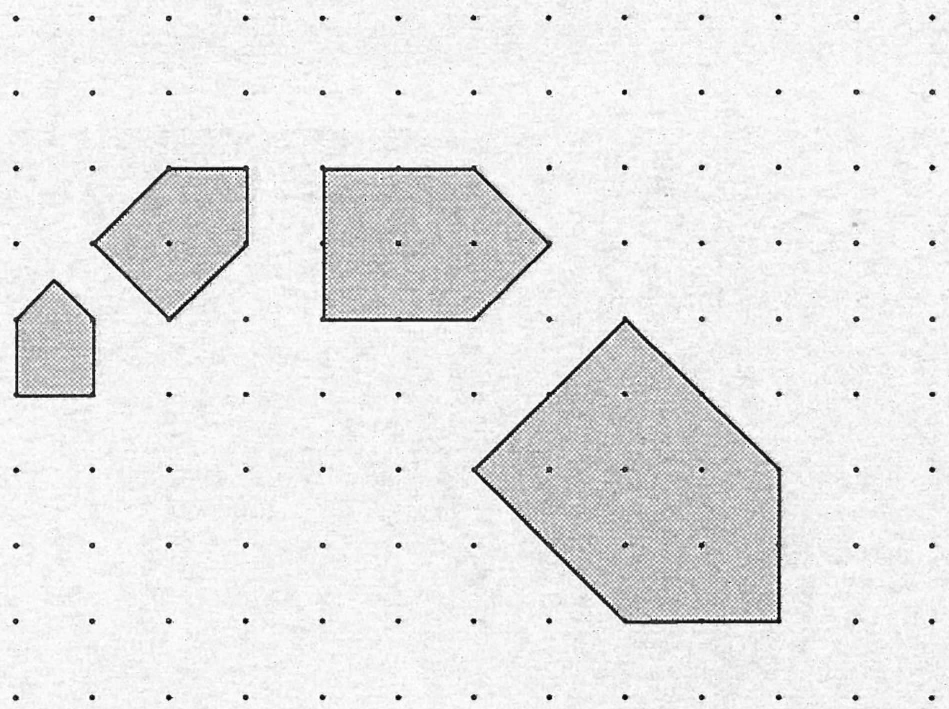
- Describe the single transformation to map:

a) L_2 on to L_0
b) L_4 on to L_0
c) L_6 on to L_0
d) L_8 on to L_0
e) L_{10} on to L_0

Use MicroSMILE *Transform* to check your work.

Shape Sequences

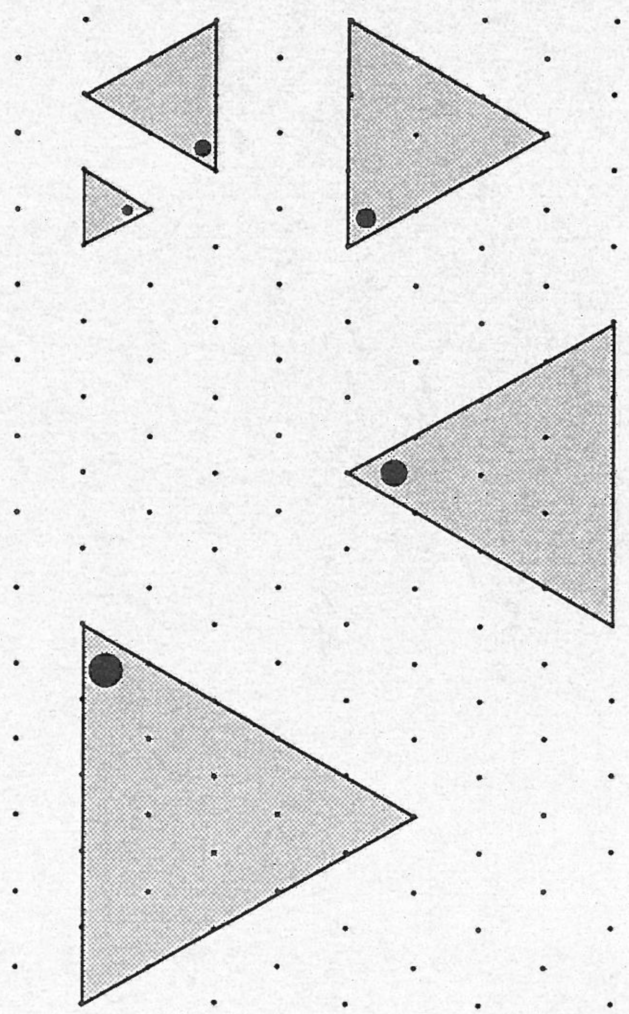
Copy these shapes on to square dotty paper.
Continue the pattern and describe any rules.



Turn over

Copy these shapes on to isometric dotty paper.

Continue the pattern and describe any rules.



Create some shape sequences of your own.

You will need: Cubes

smile

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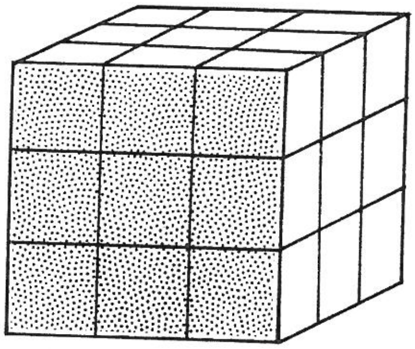
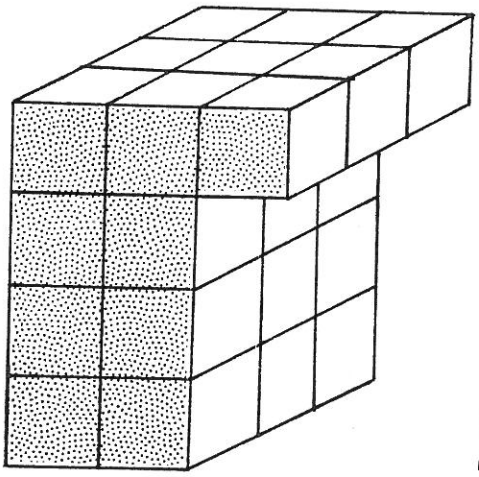
Cube Cuts



Problem: Cut the cube with a saw into centimetre cubes.
How many cuts are needed?



If you may re-arrange the pieces after each cut, can you do it with fewer cuts?

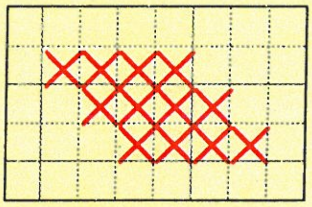
Try a 4 by 4 by 4 cube
Try other cuboids.

Cross Stitch

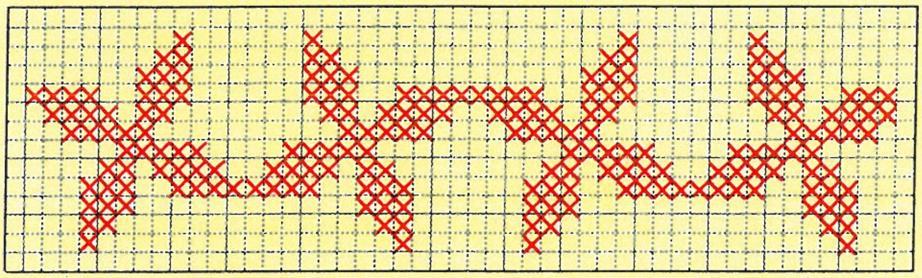
You will need 5mm square paper.

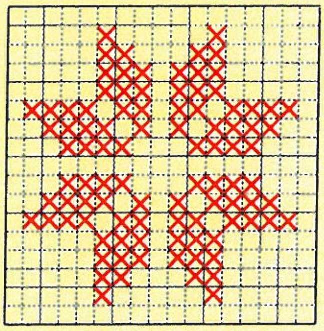
Simple motifs, using cross stitch, can be used to create elaborate patterns by using transformations including reflections, rotations and translations.

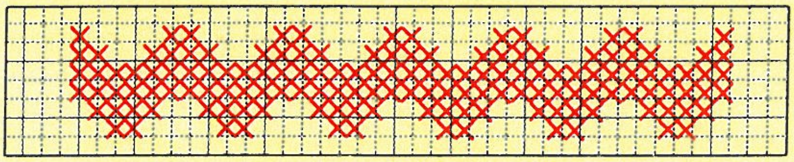
The patterns below have been created by transforming this motif.



- Analyse and describe each pattern in terms of transformations used.

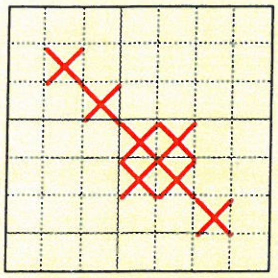




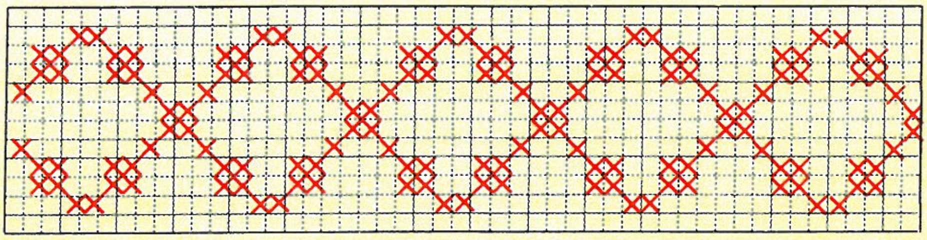


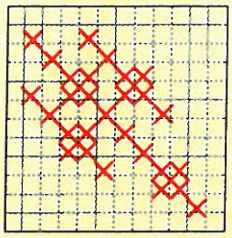
Turn over.

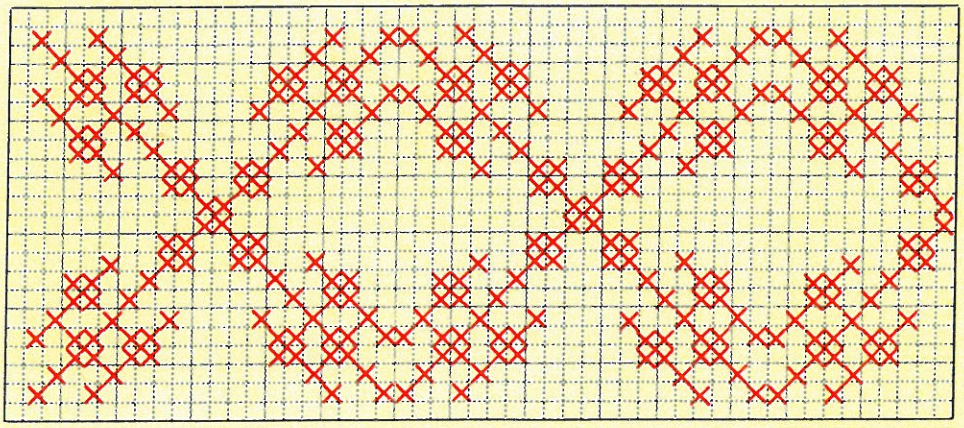
Here is another motif.



☒ Analyse and describe the patterns created.







☒ Using 5mm square paper to represent the cloth, design a motif of your own and create patterns, describing them in terms of reflections, rotations and translations.

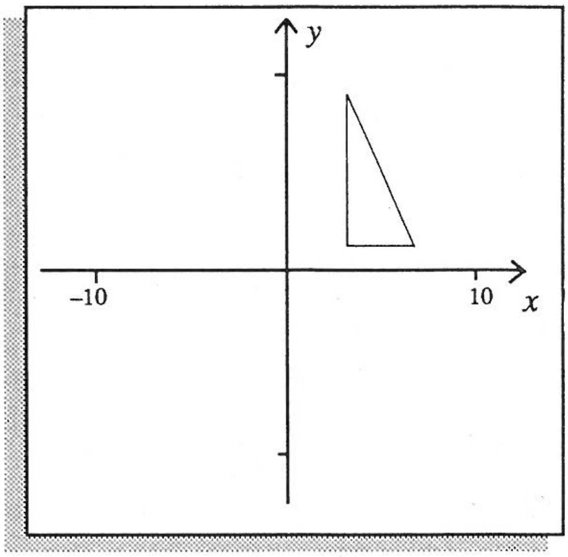
☒ Take one of your patterns and make it into a rectangular border keeping the design continuous and making right-angled bends in the appropriate places. You may find a mirror helpful to plan the corners.

You might like to choose one of your patterns and using cross stitch embroider it.

Transforming Triangles

Scale axes from -10 to +10.

Plot the points (3, 2), (6, 2) and (3, 9) and join them to form a triangle.



1) Transform the triangle using the mapping

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \end{pmatrix}$$

and describe fully the transformation.

The inverse mapping would transform the triangle to its original position. Give the inverse mapping of this transformation in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow$$

2) Describe fully the geometric transformation for each of the following mappings and give the inverse mapping for each of the transformations in the form

$$\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow$$

a) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \end{pmatrix}$

b) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ -y \end{pmatrix}$

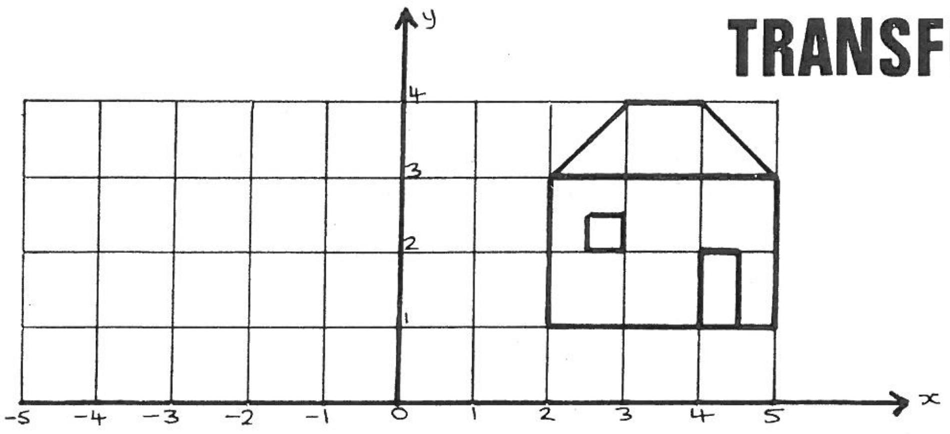
c) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \end{pmatrix}$

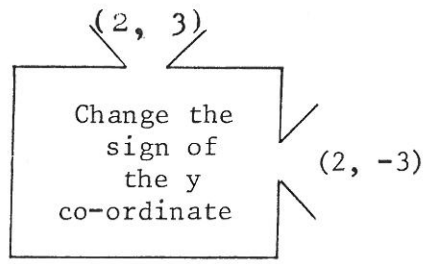
d) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ x \end{pmatrix}$

e) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ -x \end{pmatrix}$

f) $\begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow \begin{pmatrix} -y \\ -x \end{pmatrix}$

MATRICES AND TRANSFORMATIONS





This machine changes the point (2, 3) to (2, -3)

(1) Copy and complete, for the corners of the house:

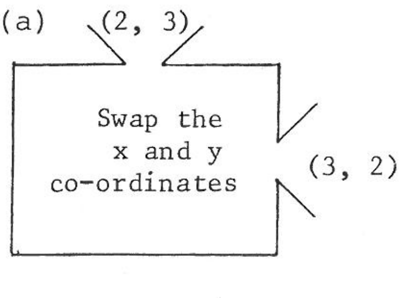
(2, 3) ➞ (2, -3)	(5, 3) ➞ (■, ■)
(3, 4) ➞ (■, ■)	(5, 1) ➞ (■, ■)
(4, 4) ➞ (■, ■)	(2, 1) ➞ (■, ■)

(2) On squared paper draw both the original house, and the house after it has been through the machine.

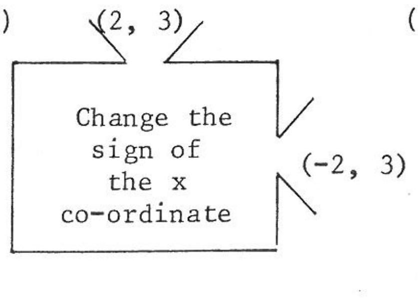
Describe what has happened to the house?

(3) Repeat (1) and (2) for each of the following machines, drawing just the new house in each case.

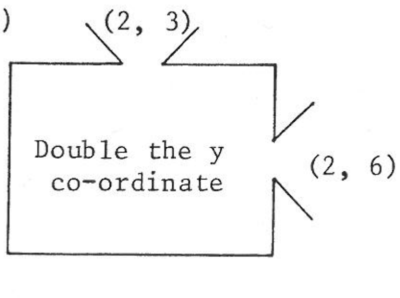
(a)



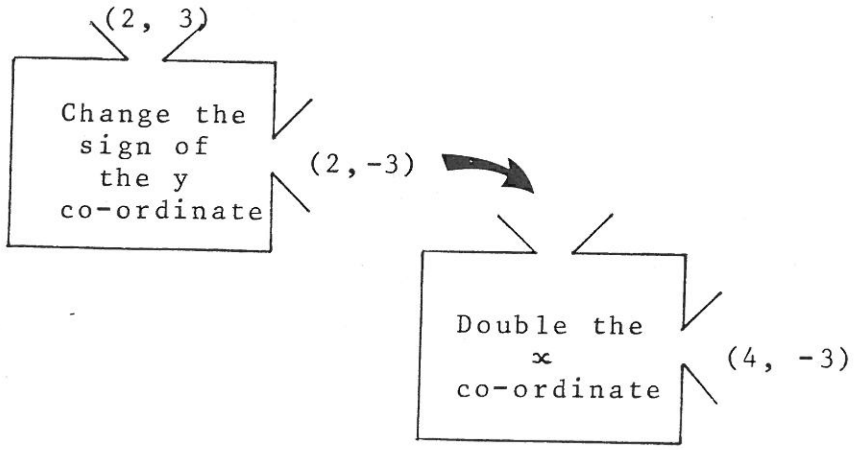
(c)



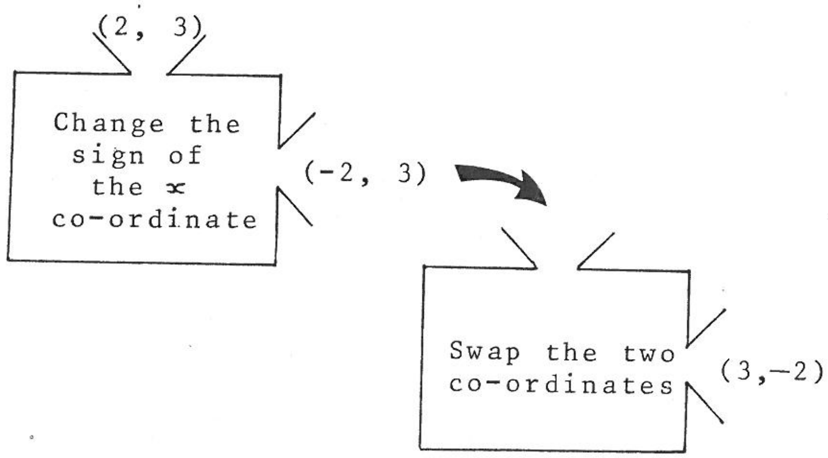
(b)



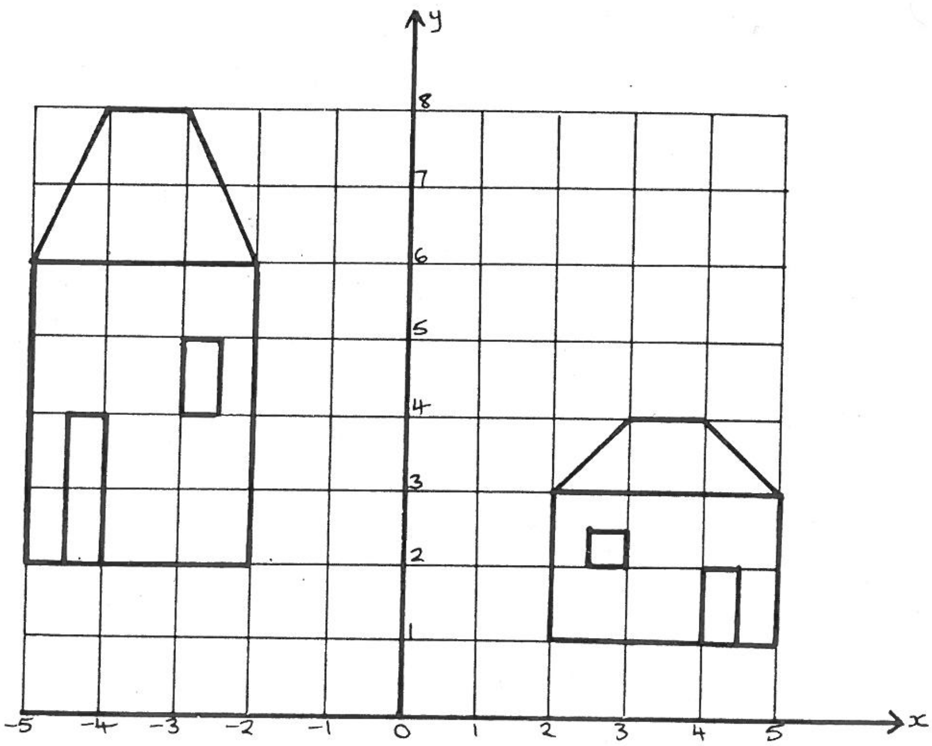
(4) Feed the corners of your original house through the <u>two</u> machines below, and draw the house you obtain.



(5) Repeat (4) for these machines:



(6) What machines are needed to make the change below? You will find it easier if you break the change into two steps.



(7)

(2, 3)

Change the sign of the y co-ordinate

(2, -3)

We can replace this machine by multiplication by a matrix. To do this the co-ordinates have to be written as a column vector:

$$\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

(a) Find what numbers must replace the stars.
(Hint: two of the stars will be 0's)

(b) Check that your matrix changes

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} \text{ to } \begin{pmatrix} 3 \\ -4 \end{pmatrix}$$

(8) Copy and complete the following to show the effect of

$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}$ on the house.

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 3 \\ 4 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 4 \\ 4 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 3 \end{pmatrix} =$$

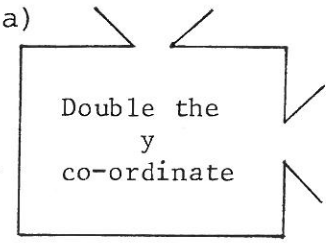
$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 5 \\ 1 \end{pmatrix} =$$

$$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} =$$

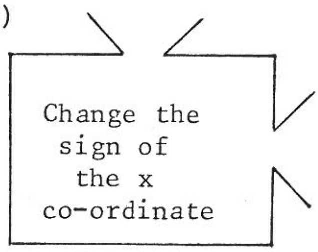
Draw the new house.

(9) Find a matrix corresponding to each of the following machines:

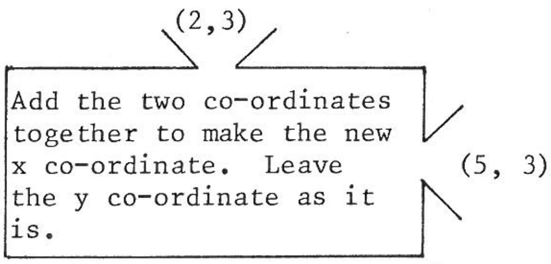
(a)



(b)



(10) a) Find what happens to the house when it is fed through this machine:

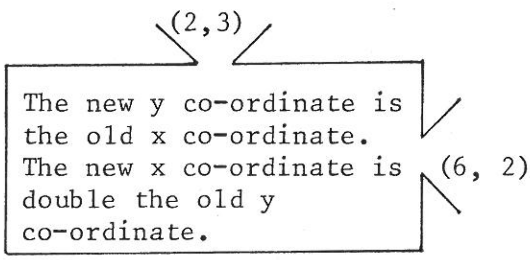


b) Draw the new house.

c) The matrix corresponding to this machine is made up of three 1's and one 0.

Find it, and check that $\begin{pmatrix} * & * \\ * & * \end{pmatrix} \begin{pmatrix} 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

(11)



a) Find what happens to the house when it is fed through this machine, and draw the new house.

b) Which of the following matrices corresponds to this machine?

$\begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$

Transformations

You may like to use MicroSMILE **Transform**.

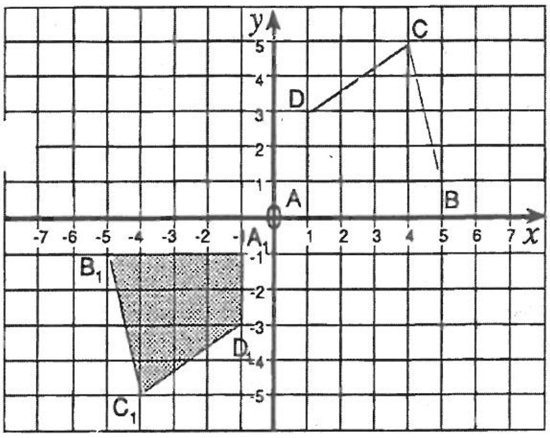
Enlargements and shears are transformations.

- Enlarging alters the area, but not the shape.
- Shearing alters the shape, but not the area.

1. Three transformations do not alter area or shape. Name them.

   *Transformations which do not alter area or shape are called **isometries**.*

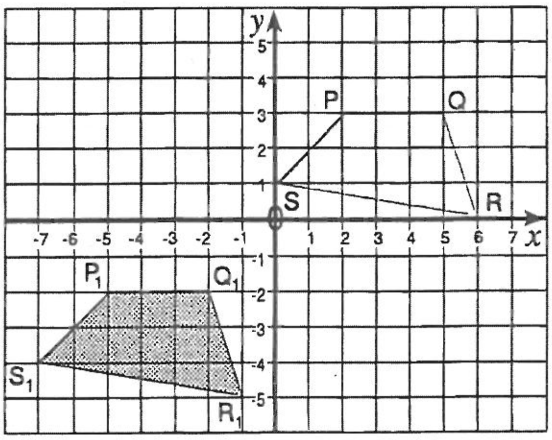
2. The quadrilateral **ABCD** can be transformed into $\mathbf{A_1B_1C_1D_1}$.



(i) a) What single transformation transforms ABCD to $A_1B_1C_1D_1$?
   b) Which two transformations could be combined to transform ABCD to $A_1B_1C_1D_1$?

(ii) What single transformation would restore $A_1B_1C_1D_1$ to the original position ABCD?

The quadrilateral **PQRS** has been transformed to $\mathbf{P_1Q_1R_1S_1}$ by transformation **K** followed by transformation **L**.



(i) What are the transformations K and L?

(ii) If PQRS were transformed by **L** followed by **K**, would the image be $\mathbf{P_1Q_1R_1S_1}$?

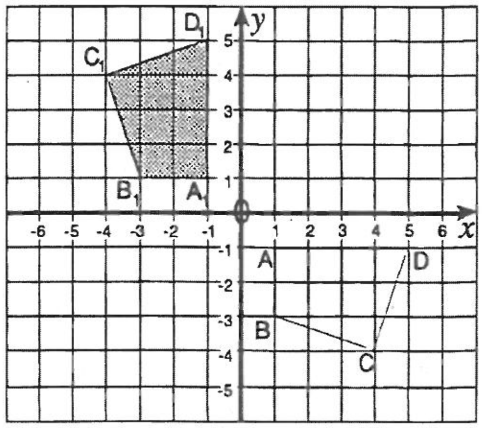
Each of the diagrams below shows a shape and its image after two transformations.

For each:

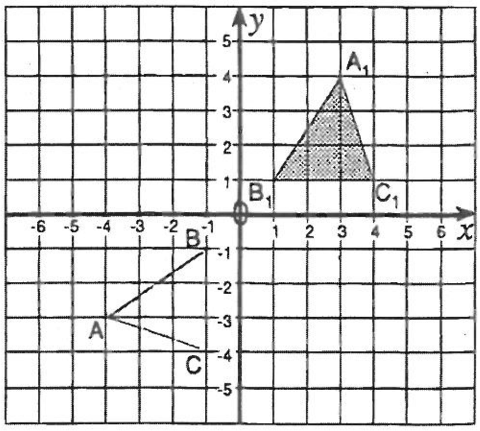
i) describe the two transformations used.

ii) describe the two transformations which would restore the image to its original position.

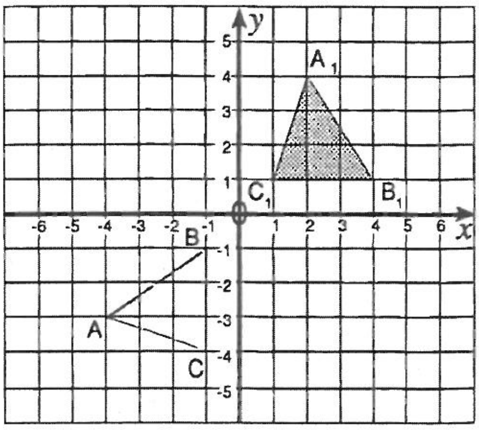
4.



5.



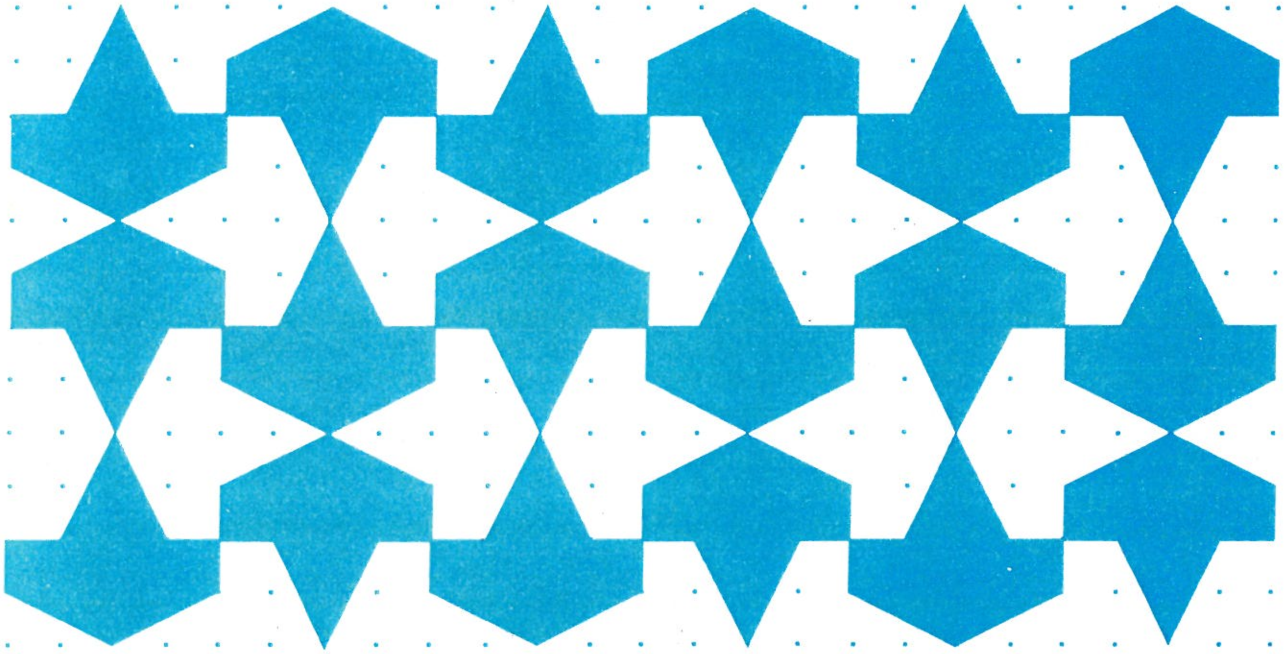
6.



Explore other combined transformations.
Summarise your findings.

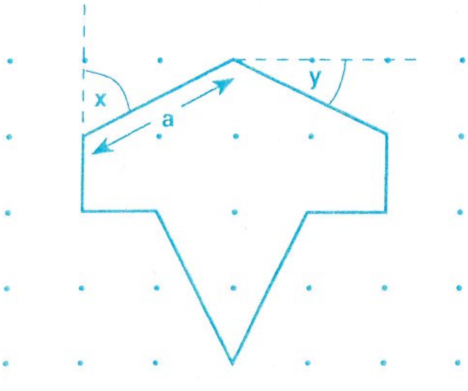
ISLAMIC Patterns in LOGO

An activity for two people.



This pattern was drawn on square centimetre dotty paper.

It is a **tessellation** of the shape below.

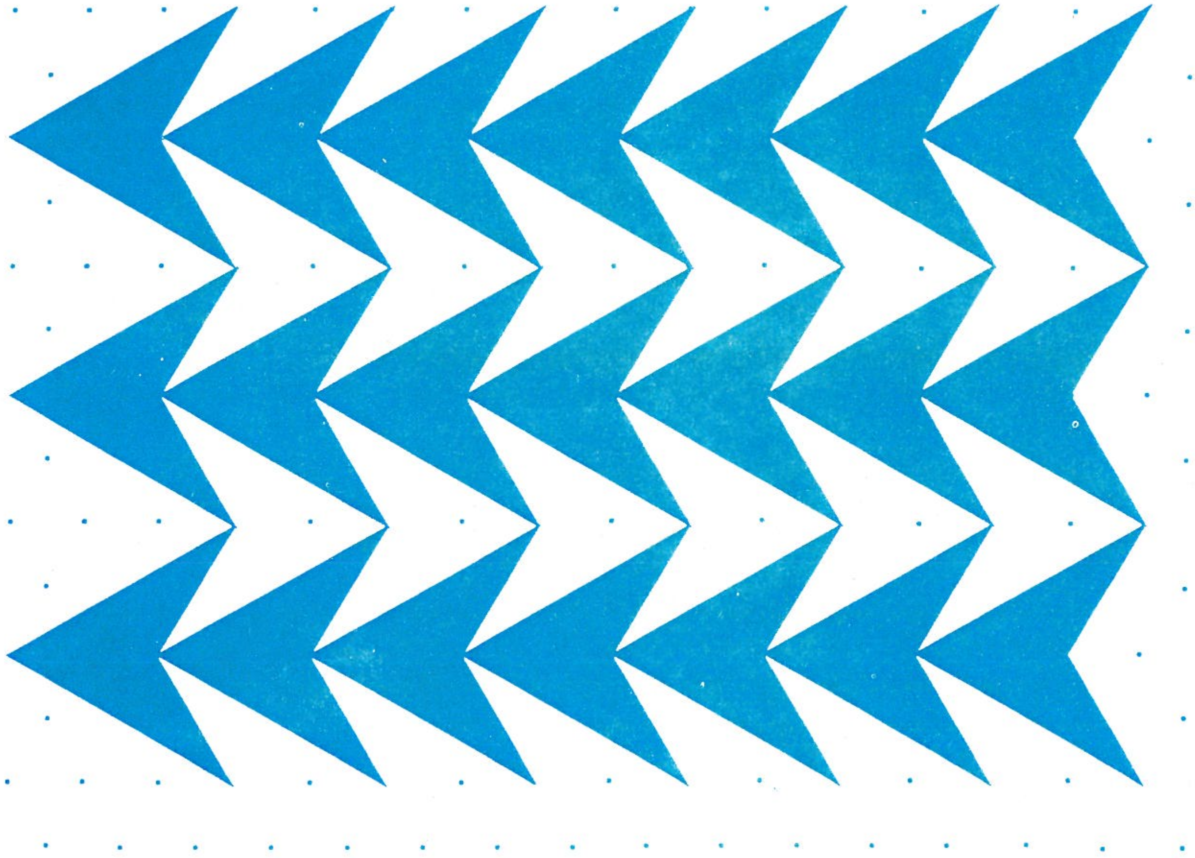


- Find **a, x** and **y** using trigonometry.
- Use your results to write a **LOGO** program to draw this tessellation.

Turn over.

The following tessellation was drawn on isometric dotty paper.

Use **LOGO** to reproduce it.



You may prefer to reproduce some other Islamic patterns using **LOGO**. *'Geometric Concepts in Islamic Art'* by *I. El-Said and A. Parman*, may give you some ideas.

Smile **1400**

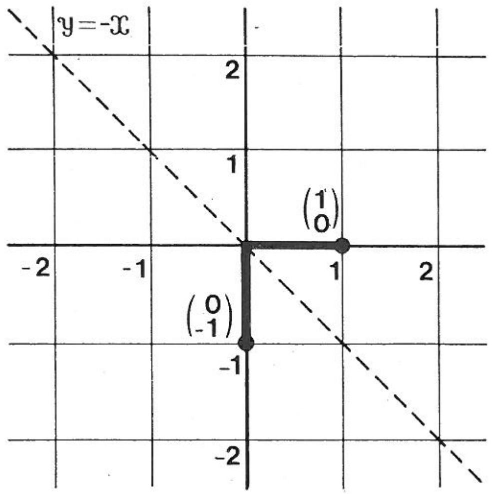
A TRANSFORMATION TECHNIQUE

This task is about finding the matrices for some transformations. There is a very simple technique which you can use. To use this technique you need to find out what happens to the unit vectors $\binom{1}{0}$ and $\binom{0}{1}$.

Follow through this example for a reflection in the line $y = -x$

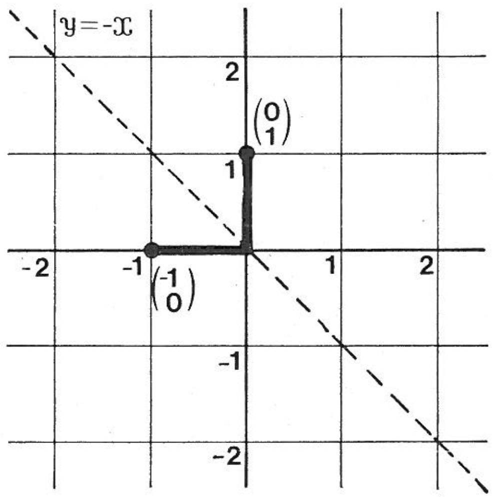
The unit vector $\begin{pmatrix}1\\0\end{pmatrix}$ is reflected to $\begin{pmatrix}0\\-1\end{pmatrix}$

$\begin{pmatrix}1\\0\end{pmatrix} \longrightarrow \begin{pmatrix}0\\-1\end{pmatrix}$



The unit vector $\begin{pmatrix}0\\1\end{pmatrix}$ is reflected to $\begin{pmatrix}-1\\0\end{pmatrix}$

$\begin{pmatrix}0\\1\end{pmatrix} \longrightarrow \begin{pmatrix}-1\\0\end{pmatrix}$



and so the matrix is : $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$

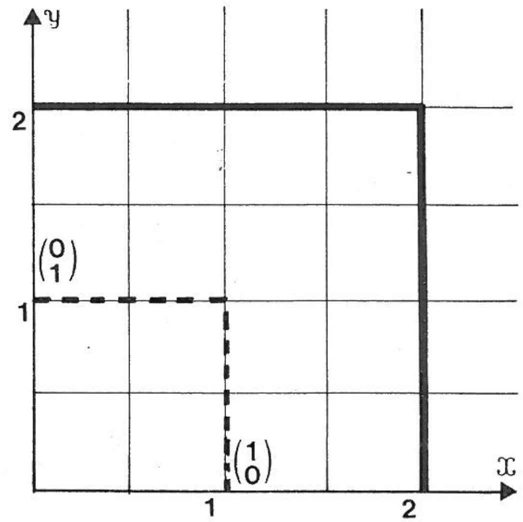
1 (a) Check that $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ is the correct matrix for the reflection in the line $y = -x$. Choose at least 6 different points to do this.

(b) Check that the matrix $\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$ reflects $\begin{pmatrix}x\\y\end{pmatrix}$ to $\begin{pmatrix}-y\\-x\end{pmatrix}$

2 (a) What happens to the position vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ after an enlargement with scale factor $^+2$ and centre of enlargement (0,0)?

(b) Write down the transformation matrix.

(c) Test your matrix with some different vectors, including $\begin{pmatrix}x\\y\end{pmatrix}$.



3 Choose some more transformations and find the matrix for each one. Try to choose simple examples so that you can draw and look at your results. Each time check that the matrix is correct.

Turn over

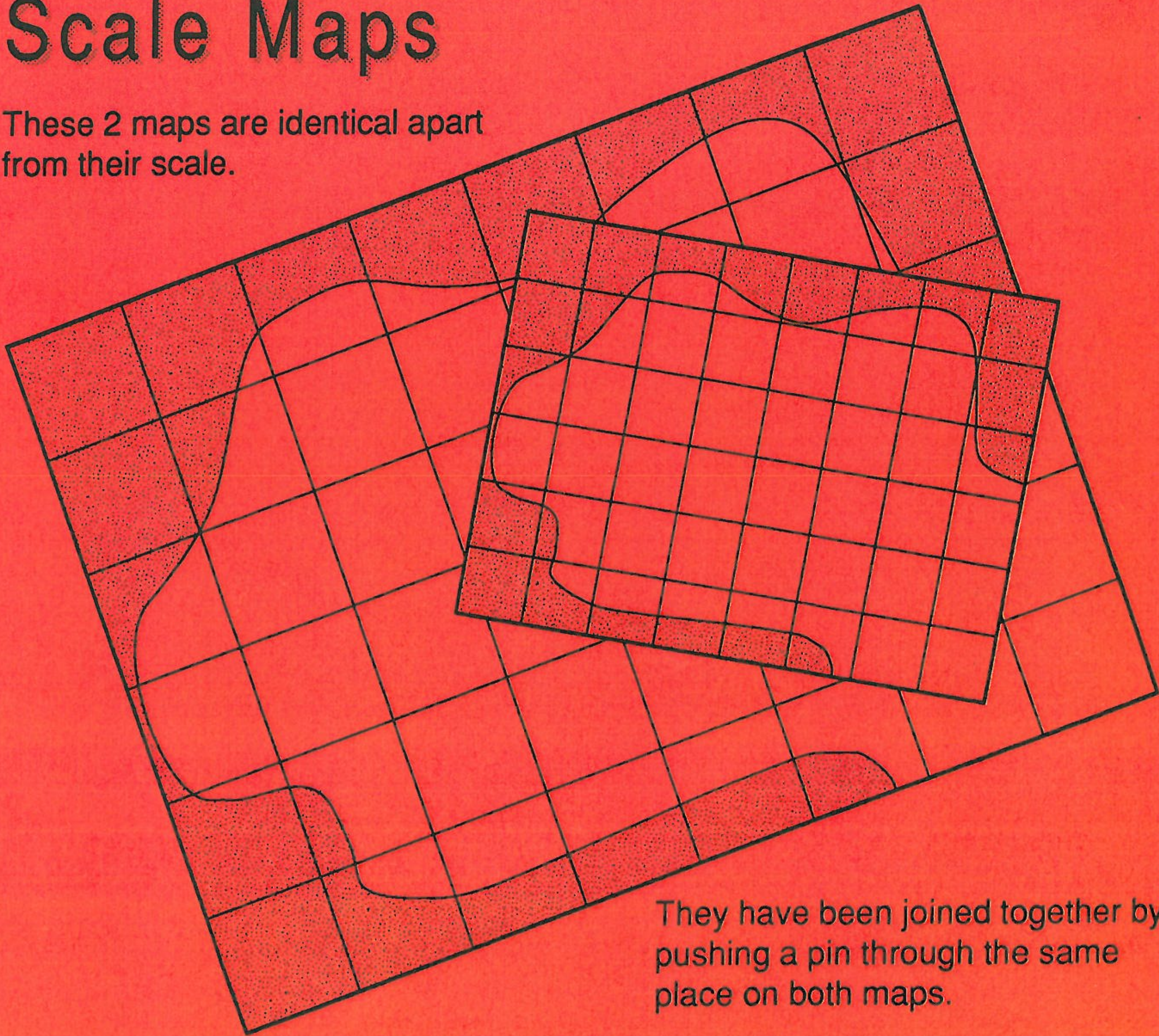
4 The work on this card has taught you a mathematical technique, but it has not explained why the technique works. The following matrix multiplications might help you understand it.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \blacksquare \\ \blacksquare \end{pmatrix}$$

Scale Maps

These 2 maps are identical apart from their scale.



They have been joined together by pushing a pin through the same place on both maps.

How can you find this point?
Explain what you did.

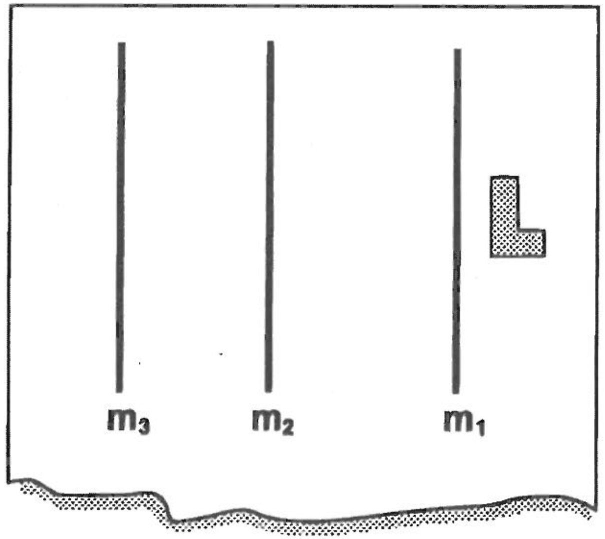
Is it still possible if one of the maps is turned through 180°?

You may like to use tracing paper.

Isometries

A. Draw an L shape on the right side of mirror line $\mathbf{m_1}$.

- Reflect it in $\mathbf{m_1}$, draw the image,
- Reflect the image in $\mathbf{m_2}$,
- Reflect the new image in $\mathbf{m_3}$.



Investigate what happens when other shapes are reflected successively in the parallel mirror lines $\mathbf{m_1}$, $\mathbf{m_2}$ and $\mathbf{m_3}$.

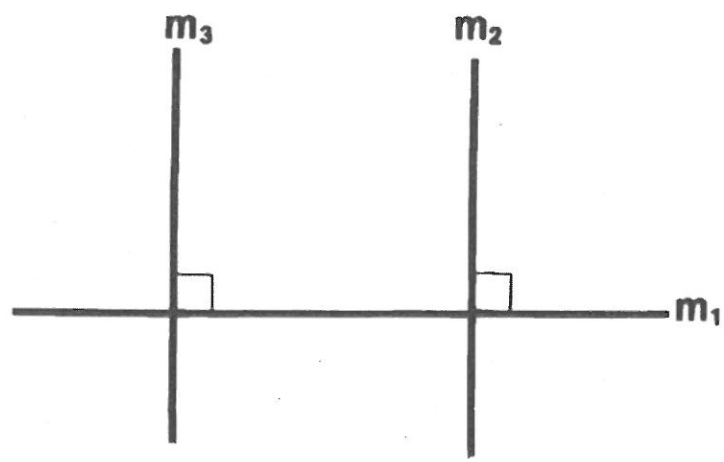
Can the resulting combined transformation be described by:

1. a translation followed by a reflection?
2. a reflection followed by a translation?
3. a single reflection?

> The combined transformation can be described by $\mathbf{M_3M_2M_1}$.
> This means reflect in $\mathbf{m_1}$, then in $\mathbf{m_2}$, then in $\mathbf{m_3}$.

B. Find the position of the single mirror that could be used to represent $\mathbf{M_3M_2M_1}$.

C. Investigate what happens when a shape is reflected successively in these three mirror lines.

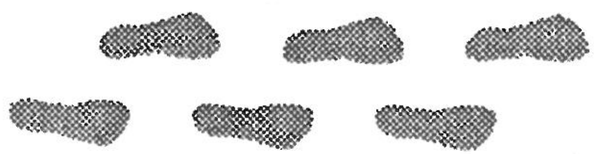


What two transformations could replace the three reflections taken in the order:

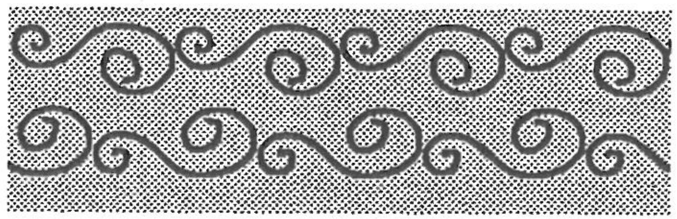
1. $M_3(M_2M_1)$?   2. $(M_3M_2)M_1$?

A **glide reflection** is the single transformation which replaces a reflection followed by a translation parallel to the mirror line. These are examples of glide reflections.

Footprints of uniform stride.



Wallpaper patterns.



Isometries are *all* transformations which *preserve shape* and *size*.
A glide reflection is an isometry, and so is a translation.

A. What two other transformations are isometries?

B. Copy and complete the table of combined transformations.

Second transformation

First transformation

followed by	**T** Translation	**M** Reflection	**R** Rotation	**G** Glide reflection
T Translation	**T**	**G** or **M**	**R**	**G**
M Reflection			**M** or **G**	**T** or **R**
R Rotation				**M** or **G**
G Glide reflection		**T** or **R**	**M** or **G**	**R** or **T**

Find at least one transformation for each of the blank spaces in the table.

Turn over.

C. Name a transformation which is an isometry when:

1. a single point only is invariant *(remains unchanged).*

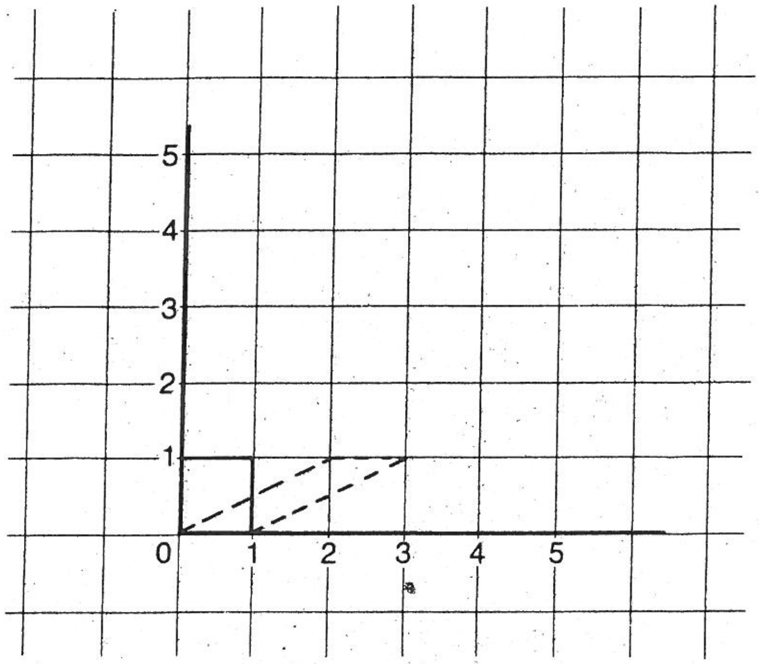
2. a single line only is invariant.

D. A transformation is represented by: $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

Draw a shape on graph paper and find out what kind of transformation it is.

E. Write down, in a matrix form similar to that in D, the glide reflection which uses the line $x = 0$ as a mirror line and a translation of +5 parallel to the line $x = 0$.

Matrices for Shears Investigation



$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$$

In this shear,

* the x-axis is invariant (it does not change)
* the point (0, 1) has moved 2 units in the x direction to the point (2, 1)
* for any point, $\dfrac{\text{distance moved in x direction}}{\text{y co-ordinate}} = 2$
* the matrix is $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$

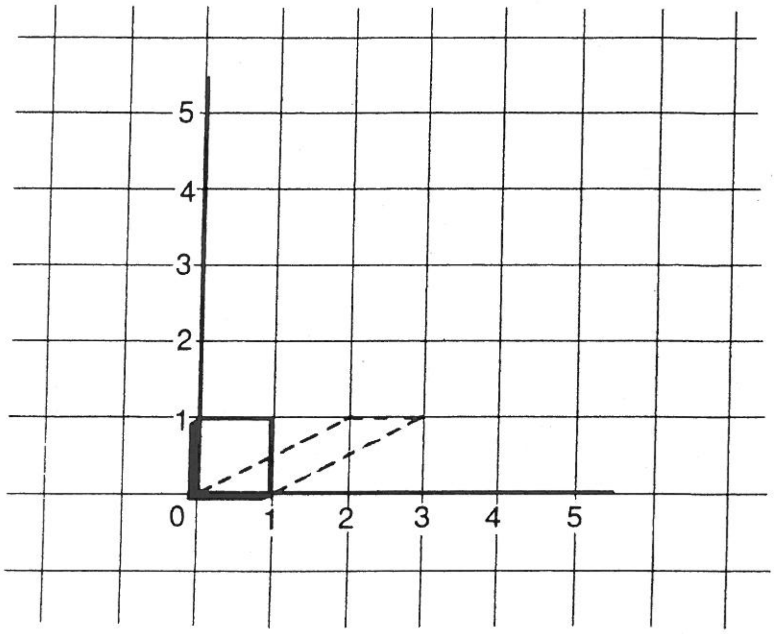
You are asked to investigate different shears and their matrices. You will need to do several shears keeping either the x-axis or the y-axis invariant. Further, you will need to record the different shears and their matrices and finally describe any connections between the shears and the numbers in the matrices.

If you can master the matrices for shears parallel to an axis, you should study shears with other invariant lines e.g. $y = 2x$

Remember the definition of a shear: every point must move parallel to the invariant line and must move a distance proportional to its distance from that line.

Turn over if you need a reminder about how to find the matrix for a particular transformation.

A useful technique to find the transformation matrix



Look at the unit vectors and see how they have been transformed.

$$\begin{pmatrix}1\\0\end{pmatrix} \rightarrow \begin{pmatrix}1\\0\end{pmatrix}$$

$$\begin{pmatrix}0\\1\end{pmatrix} \rightarrow \begin{pmatrix}2\\1\end{pmatrix}$$

Use the results to write the matrix.

$$\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$$

See 1400 *A Transformation Technique* for a fuller description.