

$$24 \times 21 = 42 \times 12$$

Lesson notes

Even though it can take a great deal of number exploration to get started with $24 \times 21 = 42 \times 12$, the *prompt* has generated sophisticated inquiries across the secondary school age range.

Students' responses to the prompt have included:

- Is the sum correct?
- An approximation shows the sum is right because $20 \times 20 = 40 \times 10$
- How do you multiply two 2-digit numbers?
- $24 + 21 = 45$ and $42 + 12 = 54$. The digits in the answers are reversed as well.
- The digits on the left and the right are all the same (1, 2, 4).
- One number on the right is half one on the left and the other is double.
- Does doubling and halving always give equal amounts in multiplication sums?
- The digits on the left have been reversed to make the numbers on the right. ("The two first digits become the second ones, and the two second ones become the two first ones.")
- Does reversing digits always work to make an 'equals sum'?
- Are there any other sums like this one?

What could we do now?

For the purpose of clarity, these ideas are presented as separate lines of inquiry even though in the classroom they are likely to overlap and one to spark another.

Put the questions into order of 'easiness'...

Under the teacher's guidance, classes have decided to order their questions. Importantly for metacognitive development, the teacher required students to justify the order on the basis of perceptions of how hard a question is to answer.

Find if the prompt is correct...

See if the digits in the 'answer' are always the same as in the 'question' (i.e. 1, 2, 4) ...

Working collaboratively, Year 7 students can quickly show the statement to be true. This is a good opportunity to ask students to model their methods to other students. Less able students might go on to make up more multiplication sums using one- or two-digit numbers. If you double and halve numbers, it is soon clear that the digits in the 'answer' are not the same as in the 'question', for example $23 \times 96 = 46 \times 48$.

Halve and double again...

$24 \times 21 = 42 \times 12 = 84 \times 6$ and 'going the other way' leads to decimals 48×10.5 . This process forms an infinite "continuing chain". A group of students might test the veracity of this statement with more examples. The

teacher could ask if the halving process ever resulted in zero. In a Year 8 class, one student expressed the 'double / half rule' algebraically as $ab = \frac{a}{2} \times 2b$. Another, having recognised that the 'double / half rule' worked independently of the 'reverse rule', conjectured (incorrectly) that "when the 'reverse rule' works, then the 'double / half rule' will work at the same time."

Change the digits by the same amount (for example, add 2 to each digit)...

This suggestion confuses the multiplicative relation under study for an additive one. There follows a brief transcript of discourse in a Year 8 lesson:

S1: Let's try 46×23 because one is half the other. *(The student should have started 26×23 using her procedure, i.e. adding two to 24 and 21, but seems to be influenced by the doubling and halving idea.)*

T: OK, reverse those...

S1: 64×32

($46 \times 23 = 64 \times 32$ now written on the board)

T: Why can't that be right?...

S2: Because both numbers on the right are bigger, so you'd get a bigger total.

S3: We could change one of the first numbers.

T: What do you mean?

S4: On one side you put one big number and one smaller.

T: Go on then. (pause)

S4: $46 \times 32 = 64 \times 23$

T (to student 1): Does that work?

S1: Yes... yes... one is halved and the other is doubled.

S5: And the digits are turned round.

S1: I was right!

T: Was she?

There then followed a lengthy discussion, heavily guided by the teacher, about why, in fact, the student's conjecture is more likely to be false because when she added two to each number, she mistakenly changed 24 to 46, not 26. As that would have left $26 \times 23 \neq 62 \times 32$, it seemed that creating more examples by adding was not feasible. The student who made the suggestion was not convinced and spent another 10 minutes trying to rescue her idea.

Use a vertical line of reflection to create more sums...

Rearranging the sum shows a line of reflection $24 \times 21 = 12 \times 42$, which leads to $96 \times 84 = 48 \times 69$ and so on.

Look for other numbers that you can double and halve and the sums 'set the digits in the other way'...

Suggestions from one mixed ability Year 8 class (below) were randomly selected until one group used the idea of digits doubling (1, 2, 4, ...). One student in particular seemed sure that the relationship between the digits was 'why it works'.

From random exploration:

$$32 \times 16 \neq 23 \times 61$$

$$24 \times 15 \neq 42 \times 51$$

$$82 \times 12 \neq 28 \times 21$$

$$28 \times 31 \neq 82 \times 13$$

The group, under the student's guidance, extended the sequence ..., 4, 8, 16 and took the digits 1, 6, 4 and 8 to form 48×16 , but,

$$48 \times 16 \neq 84 \times 61.$$

$$\text{Nor does } 84 \times 16 = 48 \times 61.$$

This approach was explained to the class when the teacher listed the examples from the period of exploration. Other groups started to pursue the same line of inquiry. 2, 4, 8 gave:

$$24 \times 28 \neq 42 \times 82$$

$$24 \times 82 \neq 42 \times 28$$

(The class missed $24 \times 84 = 42 \times 48$ and perhaps the teacher should have actively guided it in that direction.) After the class discussion, the teacher intervened with a group who, up to that moment, had played little part in the lesson. She procured 8, 16, 32 from the students, giving the digits 1, 2, 3, 6, and 8. The group was instructed to use these digits to make some 'sums like the ones on the board.' They started with 1, 3 and 6 to give:

$$13 \times 63 \neq \boxed{31} \times 36.$$

The teacher commented that 63 is slightly more than double 31. After further discussion, heavily influenced by the teacher, the group realised that two could replace the second three, to give:

$$13 \times 62 = 31 \times 26$$

More time was to elapse as they convinced themselves of the accuracy

of the statement by carrying out the calculations.

When the group presented their finding to the class, the student who had originally proposed using digits from the sequence of the powers of two exclaimed excitedly that her rule worked and came to the board to write:

$$13 \times 62 = 31 \times 26$$

After some more discussion led by the teacher, the class decided to express this observation as a rule and test it for the remainder of the lesson. The word rules were of the type: 'You double the first number to make the fourth, and double the second number to make the third.' Some students attempted to use algebra, so that $2n_1 = n_4$ and $2n_3 = n_2$ (with notation that developed out of a class discussion after the teacher drew attention to the inadequacies of 'n1'). Another group preferred to use x , y , z and n . All groups went on to look for further examples, a minority employing the algebraic equations during the search.

Use a systematic method to build more examples...

Instructions for 'building' more examples from an inquiry with Year 9 students:

1. Start with an even two-digit number (not a multiple of 10 and with different digits) less than 100 and halve it *OR* start with an odd two-digit number less than 50 and double it:

$64 \times \underline{\quad} = 32 \times \underline{\quad}$	$41 \times \underline{\quad} = 82 \times \underline{\quad}$
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2. Reverse the digits of the number on the right-hand side:

$64 \times 23 = 32 \times \underline{\quad}$	$41 \times 28 = 82 \times \underline{\quad}$
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3. Reverse the digits in the original number you thought of:

$64 \times 23 = 32 \times 46$	$41 \times 28 = 82 \times 14$
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Two pairs of students went on to show that the instructions do not always work. When choosing an even number (n), the following conditions apply: $24 \leq n \leq 86$, and half of n should not be a number in which the second digit is greater than the first (for example, 74 leads to $74 \times 73 \neq 37 \times 47$). It should be noted, however, that the second condition needs refinement because 48, among others, does lead to the acceptable $48 \times 42 = 24 \times 84$.

Use a systematic method to find more examples...

A pair of Year 7 students found $13 \times 62 = 31 \times 26$ by 'keeping the numbers low'. During a class discussion, one student's comment that 12 and 13 had now appeared led on to $14 \times 82 = 41 \times 28$ and the impossibility of using 15 within the constraints of using whole numbers only. After the teacher had

highlighted the systematic method employed, the class went on to look at 21, 22, 23 and so on. With guidance from the teacher, they found $13 \times 93 = 31 \times 39$ and with it the possibility of multiplying and dividing by three. By the end of the inquiry the class had generated an (incomplete) list of equations which students classified using ratios.

Using conjectures about the sum of the digits ...

Conjecture 1: All digits are factors of the highest digit. For example, in $13 \times 62 = 26 \times 31$, 1, 2, and 3 are factors of 6.

Conjecture 2: The sum of the digits is always a multiple of 3.

To support this conjecture a student wrote on the board:

$12 \times 42 = 21 \times 24 \dots 1 + 2 + 2 + 4 = 9$
$13 \times 62 = 31 \times 26 \dots 1 + 2 + 3 + 6 = 12$
$14 \times 82 = 41 \times 28 \dots 1 + 2 + 4 + 8 = 15$

Test conjectures and 'rules'...

Other conjectures that have arisen in classes include:

- The 'reverse rule' does not always work, for example $62 \times 14 \neq 26 \times 41$.
- If the 'reverse rule' works, then the 'double-halve rule' works simultaneously.
- To make the sides equal, both 'rules' must work at the same time.

- The ratios of the reversed pairs of numbers must be the inverse of each other.
- The ratio of numbers with reversed digits cannot be greater than 4:1.

Use algebra to find more examples...

A pair of high ability Year 8 students, working directly with a teacher, noticed, after exploration, that $ac = bd$ (where they defined a, b, c and d as the digits from the general case, i.e. $ab \times cd = ba \times dc$). This led them to substitute values for a, b, c and d . One noted that 'you need to find all the numbers that have at least four factors less than ten.' Substitution led in turn to finding examples that did not follow the doubling and halving convention. For example, 4, 3, 2 and 6 gives:

$$42 \times 36 = 24 \times 63.$$

The students, firstly, attempted to compile a complete solution set and, secondly, to categorise the equations on the basis of ratios.

Use three-digit numbers with the same constraints ...

Using the step-by-step method to generate two-digit examples, students have gone on to find three-digit numbers that satisfy the two constraints by building on the 1, 2, 4 idea:

$$\underline{124} \times 842 = 248 \times \underline{421}.$$

This has led onto other combinations of 1, 2 and 4:

$$\underline{142} \times 482 = 241 \times \underline{248}$$

$$\underline{412} \times 428 = 824 \times \underline{214}.$$

Similar groups have been found using the digits {1, 2, 3} and {2, 3, 4}. A complete solution set includes $123 \times 963 = 369 \times 321$.