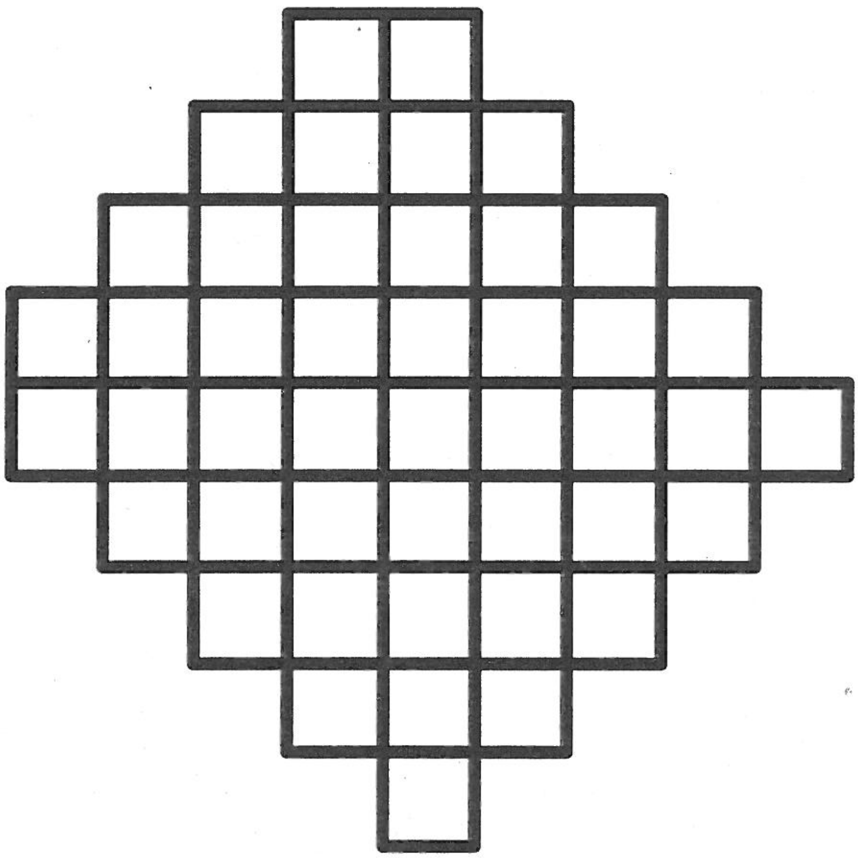
# SMILE WORKCARDS

## Patterns and Generalisations Pack Four

### Contents

# Shongo Networks

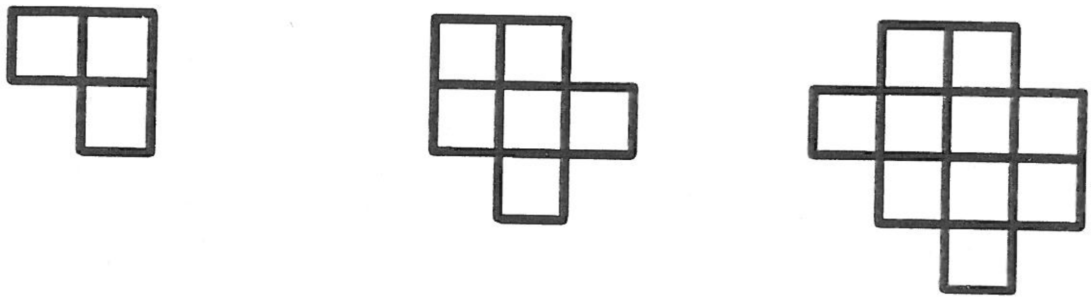
Children from the Shongo tribe in Zaire, Central Africa, draw patterns in the sand. Some of these patterns are networks like the one shown below. It has been drawn in one continuous path without going over the same line twice and always turning in the same direction. The starting position and the finishing position are different.

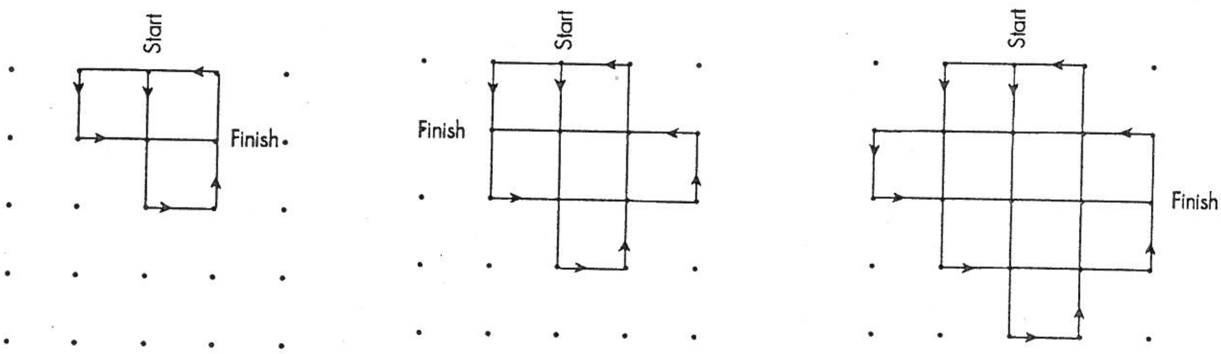


Turn over

Shongo networks can be recreated on square cm dotty paper by drawing without lifting the pencil from the paper and without going over the same line twice and by always turning in the same direction.

The first three networks are shown below.



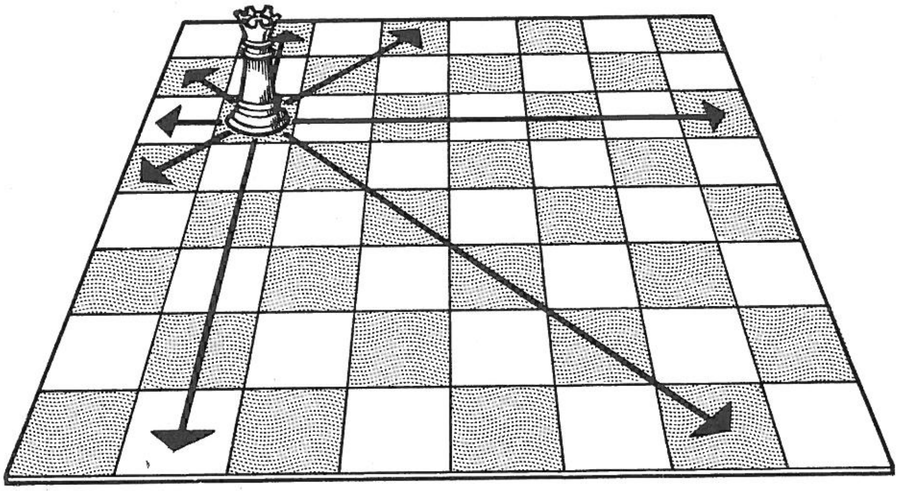



- Draw these to get the feel of how the networks are created.
- Draw some bigger ones.
- Describe the patterns you find.

LOGO can be used to create Shongo networks.

# Investigating Queens

You will find it helpful to use the micro program QUEENS.



Start with a 8 × 8 board and place one queen on it, as above. The queen is protecting 24 squares.

What is the largest number of squares which can be protected with just one queen? Where must the queen be placed?

How many squares can be protected with one queen on a 9 × 9 board? . . . a 7 × 7 board?

Investigate for boards of different shapes and sizes.

You will need: 10 counters

**smile**

# 0344

## Counter Hopping Puzzle

Put 10 counters in a row.

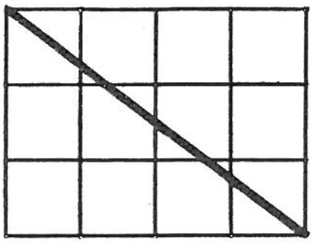
You may move any counter over the 2 nearest counters and onto the 3rd nearest.

Your first 2 moves could be the ones shown.

You must finish with 5 pairs of counters - equally spaced like this:

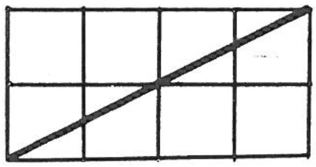
ilea

# Rectangle Diagonal



A 4 x 3 rectangle

The diagonal passes through 6 squares.



A 4 x 2 rectangle.

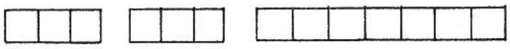
The diagonal passes through 4 squares.

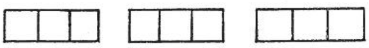
Do some more.

A table of results?............ Any rules?.............

# Threes and Sevens

Children sometimes use rods of different colours and different lengths to learn about numbers. 3-rods and 7-rods can be used to make different lengths:

 $13=(2\times3)+(1\times7)$

 $9=(3\times3)+(0\times7)$

Six different lengths *cannot* be made and the largest one not possible is 11.

*Investigate this for different pairs of rods.*

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# Number Pattern Proof

$$1 \times \frac{1}{2} = 1 - \frac{1}{2}$$

$$2 \times \frac{2}{3} = 2 - \frac{2}{3}$$

$$3 \times \frac{3}{4} = 3 - \frac{3}{4}$$

$$\vdots \qquad\qquad \vdots$$

(1) Are these equations true? Work out both sides of each one to find out.

(2) Use the pattern of numbers to write down the next three equations. Are they true?

(3) Is the 20th equation in the pattern true?

(4) To find out if the equations in this pattern are <u>always</u> true, write down the nth one:

$$n \times \frac{\blacksquare}{\blacksquare} = n - \frac{\blacksquare}{\blacksquare}$$

<u>Prove</u> that the right-hand side is equal to the left-hand side.

(5) Verify that the equation in question (4) is true when:

a) n is an integer

b) n is a fraction

c) n is negative

# Geometric Progressions

A teacher thought his class would be kept busy for the whole lesson when he asked them to add up all the powers of 3 from $3^0$ to $3^{20}$.

But, after only 2 minutes, Maria had worked out the answer to be $\frac{3^{21}-1}{2}$. The teacher was very impressed. He asked Maria to describe her method.

She explained it in this way:

$$S = 1 + 3 + 3^2 + 3^3 + \ldots + 3^{19} + 3^{20}$$

"Treble this"

$$3S = 3 + 3^2 + 3^3 + 3^4 + \ldots + 3^{20} + 3^{21}$$

"Subtract the first equation from the second."

$$3S - S = (3 + 3^2 + 3^3 + \ldots + 3^{20} + 3^{21}) - (1 + 3 + 3^2 + 3^3 + \ldots + 3^{19} + 3^{20})$$

$$2S = 3^{21} - 1$$

$$S = \frac{3^{21}-1}{2}$$

1. Use Maria's Method to find the sums of:

   (a) the powers of 2 from $2^0$ to $2^{14}$

   (b) the powers of 3 from $3^0$ to $3^{15}$

   (c) the powers of 4 from $4^0$ to $4^{15}$

   (d) the powers of 5 from $5^0$ to $5^{16}$

2. (a) Compare this series with 1(d):

   $2 + 10 + 50 + 250 + 1250 + \ldots$

   What is the sum of the first 17 terms of this series?

   (b) Which series in q.1 does this resemble?

   $3 + 6 + 12 + 24 + 48 + 96 + 192 + \ldots$

   What is the sum of the first 16 terms of this series?

   (c) Find the sum of the first 20 terms of this series:

   $2 + 6 + 18 + 54 + 162 + 486 + 1458 + \ldots$

3. If the series were put down in general terms they could all be written as:

   $a + ar + ar^2 + ar^3 + \ldots$

   Write down the values of *a* and *r* in each of the series in question 2.

*Turn over*

4. Series of the form $a + ar + ar^2 + ar^3 + \ldots$ are called *geometric progressions*.

(a) What is the 6th term of this series?

(b) What is the nth term of this series?

5. $2 + 2\times4 + 2\times4^2 + \ldots$

(a) Which is the 15th term of this series?

(b) Find the sum of the series to 15 terms.

6. Find the sums of the following series:

(a) $5 + 10 + 20 + 40 + \ldots$ to 20 terms

(b) $1 + r + r^2 + r^3 + \ldots$ to 10 terms

(c) $1 + r + r^2 + r^3 + \ldots$ to n terms

(d) $a + ar + ar^2 + \ldots$ to n terms

# Converging Sequences

**A**

| p | q |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 5 | 7 |
| 12 | 17 |
| 29 | 41 |
| . | . |
| . | . |
| . | . |

**B**

| p | q |
|---|---|
| 4 | 7 |
| 11 | 15 |
| 26 | 37 |
| 63 | 89 |
| 152 | 215 |
| . | . |
| . | . |
| . | . |

**C**

| p | q |
|---|---|
| 7 | 10 |
| 17 | 24 |
| 41 | 58 |
| 99 | 140 |
| 239 | 338 |
| . | . |
| . | . |
| . | . |

*1.* The sequences in A, B and C have been generated from the first pair of numbers. Find out how they are formed.

*Is the same rule used in each case?*

Write out the sequences and add another 5 pairs of numbers to each.

*2.* Investigate the ratios $\frac{q}{p}$ of

corresponding terms.

(e.g. for A find $\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots$
for B find $\frac{7}{4}, \frac{15}{11}, \ldots$)

*3.* Using the same rule, construct another set of ratios using two new starting numbers. What do you find?

*4.* **D**

| p | q |
|---|---|
| 1 | 1 |
| 2 | 4 |
| 6 | 10 |
| 16 | 28 |
| 44 | 76 |
| . | . |
| . | . |
| . | . |

In D the rule for column p is the same as before but the rule for column q is slightly changed

Investigate the ratios $\frac{q}{p}$ for this sequence and others of this type.

*5.* Use similar rules to create other sequences and investigate $\frac{q}{p}$ each time.

***To be clear about what is happening you will need to look at the algebra on the back of the card. Work through this before completing your work for question (5).***

*6.* Suggest a rule which, applied to two numbers x and y, will provide sequences giving ratios which tend to the value $\sqrt{n}$ where n is any number.

## Some Algebra

In answering question 1, 2 and 3 you will have found that the values of the ratios $\frac{q}{p}$ get closer and closer to 1.4142136...

To see why, look at the general pair of numbers x and y;

the next pair of numbers will be (x + y) and (2x + y).

| p | q |
| --- | --- |
| . | . |
| . | . |
| . | . |
| . | . |
| . | . |
| x | y |
| x+y | 2x+y |
| . | . |
| . | . |
| . | . |

The ratios of these pairs are $\frac{y}{x}$ and $\frac{2x+y}{x+y}$.

If the sequence is continued to more than ten terms, the ratios seem to be equal.

i.e. $\frac{y}{x} = \frac{2x+y}{x+y}$

If they are equal then

$$\begin{aligned} y(x+y) &= x(2x+y) \\ xy+y^2 &= 2x^2+xy \\ y^2 &= 2x^2 \\ \frac{y^2}{x^2} &= 2 \\ \frac{y}{x} &= \sqrt{2}\ (= 1.4142136\ldots) \end{aligned}$$

Although the ratios we calculate do not ever become equal we can assume they get closer and closer to being equal and (as we have just seen) that means they get closer and closer to $\sqrt{2}$.

What we say is, the values of the ratios **tend to the limit $\sqrt{2}$.** We call $\sqrt{2}$ **the limit** of the sequence of ratios.

The more terms we take in the sequence the more accurate is our value of $\sqrt{2}$.