For some time now we have been working with colleagues from maths, science, D&T, ICT and PE/sports on approaches to students’ physical activities which provide good opportunities for modelling in maths and science using IT tools as a catalyst. Here we share just one of the examples to give a flavour of the approach some pilot schools are now pursuing.

Thanks to the inclement weather last December many students were able to take to the slopes with all manner of toboggans. My colleague Pip has taken a video of her daughter Sophie coming downhill on her red toboggan which is 0.72m long. This was taken on a Casio Exilim digital camera at a frame rate of 30 fps and analysed using the Tracker 3 software – see Figure 1. (Tracker is a free US Open Source Physics Java applet from http://www.cabrillo.edu/~dbrown/tracker/). The trick is to keep the camera still (preferably using a tripod) and not to track or zoom to follow the moving object. It is often very helpful to set up an object of known dimensions somewhere near the centre of the scene e.g. a metre rule. Having imported the clip, you specify a known length to calibrate the picture. You can then drag and swing axes to have a suitable origin and direction. Then you can ‘track’ a point on the object while advancing the video clip and so capture data into a table. An advantage of working with video clips taken voluntarily by students away from school is that the activity takes place at their own risk and you do not have be concerned with a risk assessment!

Using the Clip Settings button the video clip has been set to run from frame 0 to frame 80, taking data from every third frame at 30 fps. The auto-tracking has successfully picked up Sophie’s foot and the front of the toboggan as target. The x- and y-coordinates are plotted on roughly equal axes in the Plots window to show how the sampled data correspond to the physical hillside. The Tracker variable $r$ measures the displacement of the object from the origin, so we can also show the displacement-time plot for comparison.
With a right-click on any Plot you can select the Analyze option and fit a regression model to the data.

[Figures 2a, 2b. Data analysed using the Tracker software]

We can export the \((t, x, y, r)\) data for further analysis and modelling in other software. For example we can copy and paste the data table to Excel, where we can do some ‘trend-line’ curve fitting, as in Figure 3. Note that for the \(y\) against \(x\) graph we need the axes to use the same scales if the snowy slope is to look realistic, as both are distances in metres. In the \(r\) against \(t\) graph this does not apply since we are plotting distances in metres against times in seconds.

[Figure 3. A screen captured from the MS Excel spreadsheet]

We can also copy and paste the data to the Spreadsheet View in the GeoGebra software for closer analysis: [http://www.geogebra.org/cms/en/download/](http://www.geogebra.org/cms/en/download//).
We can perform 2-variable Data Analysis on the B and C columns with a linear model for the $xy$-graph, and on the A and D columns with a quadratic model for the $rt$-graph.

The Input bar can be used to perform calculations in the Algebra View. The value of $m$ is just copied as the absolute value of the slope of the red line $b$ which is the linear fit to the $xy$-data. The angle $\alpha$ is computed as the angle which has $m$ as its tangent, and $ca, sa$ are the cosine and sine of $\alpha$ respectively. The value of $a$ is computed as twice the $x^2$ coefficient of the quadratic fit function $f$ for the $rt$-data. This is our estimate 1.606 ms$^{-2}$ of Sophie’s acceleration down the slope of angle 10.9°. Thus we can compute $ge = a/sa = 8.475$ ms$^{-2}$ as
our estimate of $g$, the acceleration due to gravity if there was no friction, air-resistance etc. (Here we are in effect repeating Galileo’s experiment, aka Fletcher’s trolley, for motion under gravity down an inclined plane.) However if $\mu$ is the coefficient of friction between the snow and the toboggan then, ignoring other forces, the equation for the observed acceleration is: $a = g (sa - \mu.ca)$ – from which we have our estimate of $\mu$ as $\mu = m - a/(g.ca) = 0.026$ if we substitute 9.81 for $g$. We can hide the xy-graph and concentrate on illustrating the dynamic relationships between displacement, velocity and acceleration at any time $t$.

Here we show the displacement data $(ts, rm)$ as a scatter graph together with its quadratic regression function $f$. The point $T$ has been constructed on the x-axis (which now represents time) and a perpendicular drawn to cut the graph $y = f(x)$ at $R$, where a tangent to $f$ is drawn and its slope $vt$ measured – so we have a ‘velocity measurer’ for any time $t$. We can find the corresponding velocity-time model by simple differentiation of a quadratic polynomial $f(x)$ to give the linear model $v(x)$. GeoGebra can perform both numerical and symbolic calculus (CAS). In the Input bar just enter $v(x) = \text{Derivative}[f]$ and $a(x) = \text{Derivative}[v]$ to compute the linear velocity and the constant acceleration functions. The screen shot also shows the slope of the tangent to $v$ at $V$ as the acceleration at time $t$. We can also compute the area of the quadrilateral under the velocity graph $y = v(x)$ to show the distance travelled.

Now we have modelled the motion of the toboggan we can use the results to make an animation of Sophie in action! Our graphs are in the Graphics View, in which the x-axis represents the time. We can create a second Graphics 2 View in which we will model the snowy hillside. Every geometric object can be selected to appear in either of the graphics views, or both, or neither. So we plot the xy-scattergram in Graphics 2 together with its best-fit line. We make the variable $t = x[T]$ hold the current time in seconds, and right-click on the point $T$ to turn on Animation. A little Stop/Go symbol appears in bottom-right of the view. We define the variable $r = f(t)$ as the distance travelled in time $t$ and create the point $S = (r*ca, r*sa)$ in Graphics2. Now when the start animation button is clicked the point $S$ moves down the sloping line to simulate Sophie’s motion. We just need to crop a bit of a frame from the original video clip to capture an image of Sophie on
the toboggan. Then we can fix this to the point S and scale it so that we get a realistic simulation. Finally we can create a sort of clock to show the time as the toboggan slides.

![Figure 7. Linking the graphic representations](image)

So we have brought together the four different graphic representations of the data and their models in two Graphics Views. These are linked together by the motion of the point T on the time axis of the left-hand Graphics 1 View. You can drag T manually, or make it animate. At any time you can see the values of the displacement, velocity and acceleration as well as a simulation of the toboggan coming down the hill!

![Figure 8. The force diagram for the toboggan](image)
We can also use GeoGebra to illustrate the (scaled) force diagram from which the link between the acceleration and the coefficient of friction is derived – see Figure 8. The force on the toboggan down the hill due to gravity is represented by the vector $\mathbf{v}_1$. The slider for $s$ represents the coefficient of friction and the opposing force along the hill due to friction is represented by the vector $\mathbf{v}_2$.

We chose to find models for the relationships in the data by using GeoGebra’s own least-squares regression computations. An alternative would be to use sliders to vary the coefficients for a ‘by eye’ fit for a polynomial. Figure 9 shows the idea by fitting the quadratic function $q(x) = a(x+b)^2+c$ to the $rt$-data by changing sliders for $a$, $b$ and $c$.

In this approach we are, as usual, making many assumptions, such as that the slope is straight, that air-resistance/wind is negligible etc. There is only one real source of possible data error – the calibration of the video. Can you, the reader, find out the effect on calculating the coefficient of friction $\mu$ of, say, a 10% error either way in picking up the 0.72m length of the toboggan in the video? How can we best compensate for perspective distortion of lengths between the centre and the edges of the video frame – or are such effects negligible.

The main point here is that we are not in the realm of absolutes and right answers. This is modelling with real data and, as such, is how mathematics is regularly applied to complex problems. Video capture provides a cheap, portable, reliable and visual means of collecting data in the field. Software tools like Tracker, Vernier’s Logger Pro 3, CMA Coach from the Amstel Institute, the Mathematical Toolkit, Dartfish, Silicon Coach, Kandell, Swinger Pro and Quintic all provide the means to gather data from video clips which can be exported to software such as Excel, Fathom and GeoGebra. What we hope to have shown is how the unique blend of mathematical features contained in GeoGebra can both help us perform complex calculations supporting mathematical modelling, and also provide means of displays and interactions which enable learners to understand better the close relationship between physical quantities such as displacement, velocity and acceleration.